arXiv:quant-ph/0207135 v1 23 Jul 2002

The Partition Ensemble Fallacy Fallacy

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(Dated: January 29, 2003)

The Partition Ensemble Fallacy was recently applied to claim no quantum coherence exists in coherent states produced by lasers. We show that this claim relies on an untestable belief of a particular prior distribution of absolute phase. One's choice for the prior distribution for an unobservable quantity is a matter of 'religion'. We call this principle the *Partition Ensemble Fallacy Fallacy*. Further, we show an alternative approach to construct a relative-quantity Hilbert subspace where unobservability of certain quantities is guaranteed by global conservation laws. This approach is applied to coherent states and constructs an approximate relative-phase Hilbert subspace.

PACS numbers: 03.67.Lx, 42.50.Dv, 89.70.+c

INTRODUCTION

The representation of a state and its associated interpretation are fundamental issues in quantum mechanics. For example, there are typically an infinite number of ensembles $\{P_j\}$ with which one may decompose a general state ρ via

$$\rho = \sum_j p_j P_j \ , \quad p_j \ge 0 \ , \quad P_j^2 = P_j \ . \tag{1}$$

Only when the state is pure, does this representation become unique. The laws of quantum mechanics say that (in the absence of any additional information other than the state's identity) no physical interpretation can be based on a preferred choice of an ensemble for this decomposition [1]. This result has been coined the *Partition Ensemble Fallacy* (PEF) [1].

Recently, this principle has been used to attack the meaning of coherent states in quantum mechanics [2]. It has long been argued, though without rigorous proof, that the absolute phase of an electromagnetic field is not observable [3, 4]. Thus, it has been asserted that the nominal description of light from a laser as being a coherent state $\big||\alpha|e^{-i\phi}\rangle$ should be averaged over the unknowable quantity ϕ [2]. The resulting description of the coherent state then becomes

$$\rho_{\rm PEF} = \int_0^{2\pi} \frac{d\phi}{2\pi} P(\phi) \big||\alpha|e^{-i\phi}\rangle\langle|\alpha|e^{-i\phi}\big| \tag{2}$$

$$= e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} |n\rangle\langle n| \ , \tag{3}$$

where following Ref. [2] we have taken the prior probability $P(\phi)$ to be flat. Thus, the ensemble of states being produced by a laser could as easily be chosen as number states instead of coherent states with fixed coherent amplitude $|\alpha|$. Recalling that the PEF disallows interpretations for states based on a preferred choice, we should infer that experiments using lasers cannot be reliably interpreted as demonstrating features or properties of coherent states. This is the logic behind the argument of Rudolph and Sanders [2].

The automatic assumption that the prior distribution of phases $P(\phi)$ should be taken as flat appears straightforward. Ordinarily, when one has an unknown quantity, one assigns a prior distribution based on whatever prior information is available. If one lacks *any* information then one tries to rely on symmetries in the problem. Thus, since any choice of absolute phase ϕ leads to the same observable results the flat prior appears to be the canonical choice. However, there is something fishy about this reasoning. That a prior distribution is a meaningful summary of our knowledge (or lack thereof) depends on the full procedure of inference (for example, Bayesian). Here we are talking about a quantity which is not simply unknown, but unknowable. No inference can ever be made about it based on new information. In fact, *absolutely any* choice can be made for $P(\phi)$ and that choice is completely untestable. All predictions one could make would agree whatever choice one had for $P(\phi)$. To this extent, one's choice for the prior distribution for an unobservable quantity is a matter of 'religion.' It lies outside the realm of science.

By contrast to the choice for a prior made by Rudolph and Sanders, a choice consisting of a delta function would make the entire application of the PEF inadmissible (since we would be dealing with pure states). However, since the application of a principle cannot depend upon an arbitrary (and untestable) choice it clearly follows that the PEF does not apply here. We call this principle the *Partition Ensemble Fallacy Fallacy* (PEFF).

If the absolute phase is truly unobservable then the PEFF guarantees experimentalists the freedom (of 'religion') to continue talking about coherent states as a state of the form of (2). Even if one prepares a known state (3) as if to be the laser output, a coherent state is still applicable. The PEFF simply says that the usual coherent state language is unfalsifiable. Mathematically, the freedom to choose the prior due to the unobservability of ϕ induces an equivalent relation between all states (2).

This indicates that this freedom of 'language' imposes the absence of the physical meaning of purity (or mixture) on the state representation. This suggests that a single mode state fails to represent the quantum nature of the system due to the unobservability of absolute phase.

Since this argument relies on the unproved unobservability of ϕ, we may consider the validity of the coherent-state language in the absence of the unobservability of absolute phase. As we have seen, as long as we deal with a single mode state, the free choice of 'language' always exists. This implies that to construct a state which is free from the prior, the single mode representation has to be abandoned. As result, the difference between the unobservability of absolute phase and the untestable prior would not make a big difference in the requirement for state representation in quantum mechanics. The state representation for the laser output might have to involve multimode to serve the quantum nature of the system. In this paper we investigate this in detail.

We start by considering much simpler situations where variables are guaranteed to be unobservable due to global symmetries and the Wigner-Araki-Yanase (WAY) theorem. We shall see that reasoning of Rudolph and Sanders would leave most of the formalism of quantum mechanics as unavailable if generally applied to unobservable quantities. Typically, in these cases, an alternative approach based on relative variables allows one to circumvent the entire discussion about unobserved quantities. It appears impossible to construct an exact relative-phase Hilbert subspace, however we give an explicit construction of an approximate relative-phase Hilbert space. The advantage of this formulation, despite the added complication, is that it allows the usual coherent state language to be used without assuming that phase is unobservable.

THE WIGNER-ARAKI-YANASE THEOREM

The Wigner-Araki-Yanase (WAY) theorem gives us a playground of systems where unobservability of certain quantities is guaranteed by global conservation laws. The WAY theorem states that any operator which does not commute with an operator of the global conservation is not observable [6, 7]. Consider a system consists of the observed subsystem and its measuring apparatus. As the total momentum $\hat{\Pi}$ of the system is conserved, a position operator $\hat{x}$ of the observed subsystem is unobservable. This is because the position operator does not commute with the total momentum and such measurement process violates the conservation law. Application of Rudolph and Sanders reasoning to this example results in an arbitrary position eigenstate to be

$$\int dX P(X)\hat{D}(X)|x\rangle\langle x|\hat{D}^{\dagger}(X) = \int dp|p\rangle\langle p| = \hat{1}, \quad (4)$$

where $\hat{D} = e^{-iX\hat{\Pi}}$ is the position displacement operator.

As the position $\hat{x}$ is unobservable and hence $P(X)$ is completely arbitrary, all states of the left hand side of (4) are equivalent. The freedom to choose the prior is to be shown by expectation values of all possible observables. The total momentum conservation restricts Hamiltonian and time evolution unitary operator to commute with the total momentum, that is $\hat{U}\hat{\Pi}\hat{U}^{\dagger} = \hat{\Pi}$. The expectation of an arbitrary observable $\langle\hat{A}\rangle$ is given by

$$\begin{aligned}\langle\hat{A}\rangle &= \mathrm{Tr}[\hat{A}\hat{U}^{\dagger}\rho\hat{U}] \\ &= \int P(X)dX\langle x|\hat{U}\hat{A}\hat{U}^{\dagger}|x\rangle. \\ &= \langle\phi|\hat{U}\hat{A}\hat{U}^{\dagger}|\phi\rangle.\end{aligned} \quad (5)$$

The prior probability distribution $P(X)$ is completely arbitrary for any physical quantities associated with the system. Therefore we can use the pure-state language to consistently treat the system.

RELATIVE QUANTITIES AND STATE PREPARATION

Now we give a way to construct relative-quantity Hilbert space. According to the WAY theorem, a relative quantity of the unobservable absolute operators can be observable. For example, a relative position operator of the observed system to the apparatus $\hat{x}_1 - \hat{x}_2$ $(= \hat{x}_r)$ commutes with the total momentum and hence is observable, where $\hat{x}_1$ and $\hat{x}_2$ are the absolute positions of the observed system and the apparatus respectively. We take eigenstates of an operator $\hat{x}_a = \hat{x}_1 + \hat{x}_2$ to construct the entire Hilbert space together with eigenstates of $\hat{x}_r$. The Hilbert space for the entire system (the observed system and the apparatus) can be expanded by $\{|x_r\rangle \otimes |x_a\rangle\}$ as well as by $\{|x_1\rangle \otimes |x_2\rangle\}$. To construct a relative-position Hilbert space, we start with separable states given by

$$|\psi\rangle = |\psi_r\rangle \otimes |\psi_a\rangle, \text{ where } \left\{ \begin{array}{l} |\psi_r\rangle = \int dx_r \psi_r(x_r)|x_r\rangle \\ |\psi_a\rangle = \int dx_a \psi_a(x_a)|x_a\rangle. \end{array} \right. \quad (6)$$

Here the state is separable in terms of the two subspaces of $\{|x_r\rangle\}$ and $\{|x_a\rangle\}$.

Similarly to the procedure of Eq. (4), integrating the state over the amount of displacement X by the operator $\hat{D}(X)$ with the prior distribution $P(X)$, we obtain

$$\rho_{ra} = \int dX P(X) e^{-iX\hat{\Pi}}|\psi\rangle\langle\psi|e^{iX\hat{\Pi}}. \quad (7)$$

However, the operator $\hat{x}_r$ commutes with the total momentum Π, then the state $|\psi_r\rangle$ is preserved under the action of the displacement operator. This allows the density matrix to be

$$\rho = |\psi_r\rangle\langle\psi_r| \otimes \rho_a. \quad (8)$$

where

$$
\begin{aligned}
|\psi_r\rangle &= \int dx_r \psi_r(x_r)|x_r\rangle \\
\rho_a &= \int\int\int dXP(X)dx_a dx'_a \psi_a(x_a)\psi^*_a(x'_a) \\
&\quad \times e^{-iX\hat{\Pi}}|x_a\rangle\langle x'_a|e^{iX\hat{\Pi}}.
\end{aligned}
$$

The state $|\psi_r\rangle$ is on the relative-position Hilbert space. The relative-position Hilbert space is constructed to be completely free from the prior distribution and the argument associated with the unobservability.

By assuming arbitrary separable states (6), we naturally include the case of non-separable states. Some entangling operators such as a SUM gate $(\exp\big(-i\hat{x}_r\otimes\hat{\Pi}\big))$ commute with the total momentum and hence are allowed. These operators with superpositions can generate entanglement. In this case, the state has to be more generally represented by

$$
|\phi\rangle = \int dx_r dx_a \psi(x_r, x_a)|x_r, x_a\rangle. \tag{9}
$$

In the case where the state is entangled, the function $\psi(x_r, x_a)$ cannot be written as $\psi_r(x_r)\psi_a(x_a)$. If we take the same procedure to this state, then we have

$$
\begin{aligned}
\rho &= \int\cdots\int dx_a dx_{a'} dx_r dx_{r'} \psi(x_r, x_a)\psi^*(x'_r, x'_a)|x_r\rangle\langle x'_r| \\
&\otimes \int dXP(X)|x_a + X\rangle\langle x'_a + X|.
\end{aligned} \tag{10}
$$

It is unfortunately not trivial how a relative-position Hilbert space can be extracted in this state. Next we will see a consideration of state preparation under the conservation laws helps us to construct a consistent relative-position Hilbert space.

As the total momentum is constant, any eigenstate of the total momentum can be a state of the total-momentum Hilbert space. A superposition of the total momentum eigenstates is also consistent with the constant total momentum requirement. As we have discussed above, a superposition can generate entanglement with some entangling operator, while an eigenstate of the total momentum cannot be entangled with the relative-position subspace. With an eigenstate of the total momentum, none of the operators which generate a superposition of the eigenstates is allowed under the conservation law. This leads to the necessity of a third system to be involved in the state preparation process. When we consider the whole process of measurement including state preparation, each eigenstate of the total momentum is solely consistent with the conservation law. Furthermore, it is inconsistent to treat the two processes, a measurement and a state preparation, in different spaces. This means that even if the system of the observe system and the apparatus recovers the conservation of the total momentum after the state preparation, the system cannot completely eliminate the third system. A closed system with the momentum conservation is invariant in transformation by its absolute position, so different values of the total momentum gives the same state to the system. Two different values of the *total momentum* $\hat{\Pi}$ become distinct when these are realized in the extended system. Thus, the superposition should be considered to lie on a relative-quantity subspace in the extended system. For a closed system with the momentum conservation, as the eigenstate of the total momentum is the only state consistent, any state can be represented as (6) and hence the relative-position subspace always can be constructed.

PHASE OF LASER LIGHT FIELD

Now we turn our attention back to the laser light field and apply our procedure to this particular case. The expected unobservability of the absolute phase is the motivation to introduce a relative phase of two mode coherent state given by

$$
\begin{aligned}
|\alpha, \beta\rangle &= \big|\,|\alpha|e^{-i\phi_\alpha}\big\rangle \otimes \big|\,|\beta|e^{-i\phi_\beta}\big\rangle \\
&= \sum_{n_1}^{\infty}\sum_{n_2}^{\infty} \frac{\alpha^{n_1}\beta^{n_2}}{\sqrt{n_1!n_2!}}|n_1, n_2\rangle.
\end{aligned} \tag{11}
$$

The total photon number of the state is $N = n_1 + n_2$ and the difference photon number is $M = (n_1 - n_2)/2$ which is either integer (for even total photon numbers) or half-integer (for odd total photon numbers). The state (11) can be alternatively expanded by the eigenstates characterized by these quantum numbers N and M as

$$
\begin{aligned}
|\alpha, \beta\rangle &= e^{-\frac{|\alpha|^2+|\beta|^2}{2}} \sum_{N=0}^{\infty}\sum_{M=-N/2}^{N/2} \alpha^{\frac{N}{2}+M}\beta^{\frac{N}{2}-M} \\
&\times \Big[(\frac{N}{2}+M)!(\frac{N}{2}-M)!\Big]^{-1/2}|N, M\rangle.
\end{aligned} \tag{12}
$$

Obviously this state is not separable in terms of the two subspaces, $\{|N\rangle\}$ and $\{|M\rangle\}$. In this case we cannot simply extract the relative-phase subspace, so we require an ingredient to approximately extract the relative-phase subspace. Taking a set of parameters as

$$
\begin{cases}
\frac{|\alpha|}{\langle\hat{N}\rangle^{1/2}} = -\sin\frac{\theta}{2},\ \frac{|\beta|}{\langle\hat{N}\rangle^{1/2}} = \cos\frac{\theta}{2} \\
\phi_\alpha - \phi_\beta = \phi_r.
\end{cases} \tag{13}
$$

The two mode coherent state can be written as the sum of spin coherent states, yielding

$$
|\alpha, \beta\rangle = e^{-\frac{\langle\hat{N}\rangle}{2}} \sum_{N=0}^{\infty} \frac{\big(\langle\hat{N}\rangle^{1/2}e^{-i\phi_\beta}\big)^N}{\sqrt{N!}}|N, \theta, \phi_r\rangle. \tag{14}
$$

Here $|N, \theta, \phi_r\rangle$ is a spin-$N/2$ coherent state with the parameterization (13). Alternatively the spin coherent state may be parameterized by ξ $(= -\frac{|\alpha|}{|\beta|}e^{-i\phi_r})$ as

$$|N,\xi\rangle = \sum_{M=-\frac{N}{2}}^{\frac{N}{2}} \binom{N}{\frac{N}{2}-M}^{\frac{1}{2}} (1+|\xi|^2)^{-\frac{N}{2}} \xi^{\frac{N}{2}+M} |N,M\rangle \quad (15)$$

If we ignore the spin coherent space, then the state for the total photon number can be realized as a coherent state of $|\langle\hat{N}\rangle^{1/2}e^{-i\phi_\beta}\rangle$. When $\langle\hat{N}\rangle^{1/2}$ goes infinity, the contribution of components for small N to the sum is negligible and the main contribution is the terms of the order $N \sim \langle\hat{N}\rangle^{1/2}$. In the large limit of N, the spin coherent state can be contracted to a Weyl-Heisenberg (WH) coherent state. When $|\alpha| \sim |\beta|$, the state can be typically contracted to a WH coherent state,

$$|\theta,\phi_r\rangle \rightarrow |-\sqrt{2}|\alpha|e^{-i\phi_r}\rangle. \quad (16)$$

At the limit, this coherent state is approximately separable with the subspace of the total photon number, hence an approximate relative-phase subspace has been constructed. However this approximation is not so useful as the limit brings $|\alpha|$ also to infinity as $\sqrt{2}\alpha \sim \langle\hat{N}\rangle^{1/2}$. By contrast, when $|\alpha| << |\beta|$ is satisfied, the group contraction may be taken in the order of $\langle\hat{N}\rangle$. In this case the spin state $|N,\theta,\phi_r\rangle$ is contracted by a parameter $\epsilon = 1/|\beta|$ as

$$\xi = -\epsilon|\alpha|e^{-i\phi_r},\ (\epsilon \rightarrow \infty). \quad (17)$$

In this contraction, the spin size given by $|\beta|^2$ goes to infinity with $\epsilon \rightarrow 0$ and the state is contracted to a WH coherent state $|-|\alpha|e^{-i\phi_r}\rangle$.

The coherent state from laser can be approximately represent as

$$|\alpha,\beta\rangle\langle\alpha,\beta| \sim |-|\alpha|e^{-i\phi_r}\rangle\langle-|\alpha|e^{-i\phi_r}| \otimes \rho_N, \quad (18)$$

under the condition

$$\langle\hat{N}\rangle \sim |\beta|^2 >> |\alpha|^2. \quad (19)$$

The coherent state is constructed in the subspace of the relative phase.

To conclude, we have shown the explicit construction of an approximate relative-phase Hilbert space. The two mode coherent state can be represented as a pure coherent state in the relative-phase subspace under the condition (19). This state presentation of relative phase is free from a choice of prior distribution, and hence circumvents the entire discussion about unknowable absolute phase.

SLB thanks Asher Peres and Howard Wiseman for discussions. This work is funded in part under project QUICOV as part of the IST-FET-QJPC programme.

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